JHEP HYPER VERSION

Received: *September 11, 2000*, Accepted: *October 11, 2000*

Dynamics of strings in noncommutative gauge theory

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Abstract: We continue our study of solitons in noncommutative gauge theories and present an extremely simple BPS solution of $\mathcal{N} = 4$ U(1) noncommutative gauge theory in 4 dimensions, which describes N infinite D1-strings that pierce a D3-brane at various points, in the presence of a background B-field in the Seiberg-Witten $\alpha' \to 0$ limit. We call this solution the N-*fluxon*. For $N = 1$ we calculate the complete spectrum of small fluctuations about the fluxon and find three kinds of modes: the fluctuations of the superstring in 10 dimensions arising from fundamental strings attached to the D1 strings, the ordinary particles of the gauge theory in 4 dimensions and a set of states with discrete spectrum, localized at the intersection point — corresponding to fundamental strings stretched between the D1-string and the D3-brane. We discuss the fluctuations about the N-fluxon as well and derive explicit expressions for the amplitudes of interactions between these various modes. We show that translations in noncommutative gauge theories are equivalent to gauge transformations (plus a constant shift of the gauge field) and discuss the implications for the translational zero modes of our solitons. We also find the dyonic versions of N-fluxon, as well as of our previous string-monopole solution.

Keywords: Solitons Monopoles and Instantons, Brane Dynamics in Gauge Theories, Non-Commutative Geometry.

JHEP10(2000)021

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1. Introduction

Field theories on noncommutative spaces [1, 2] emerge as limits of M theory compactifications [3] or of string theory with D-branes in the presence of a background Neveu-Schwarz B-field [4, 5, 6]. The interest in such theories is motivated by many analogies between noncommutative gauge theories and large N ordinary non-abelian gauge theories [7, 8], and by the many features that noncommutative field theories share with open string theory [9, 8, 10].

In this paper we continue the study [11] of non-perturbative dynamical objects in noncommutative gauge theory, specifically four dimensional gauge theories with an adjoint scalar field. Our paper is organized as follows.

In section 2 we briefly review the setup of noncommutative gauge theory. In [11] we discussed some general features of these theories and generalized Nahm's equations for BPS solutions of the classical field equations to the noncommutative theory. We solved these equations for the analogue of a single monopole for noncommutative U(1) theory. The solution we constructed was nonsingular and sourceless, and described a smeared monopole connected to a string-like flux tube. We interpreted this string-monopole as the reflection of a D1 string attached to the D3-brane in the presence of a background Neveu-Schwarz B-field. We calculated the tension of the string and found precise agreement with that expected from the D1-string. In section 3 we briefly review this solution and then by deforming it construct an extremely simple classical BPS solution of noncommutative U(1) gauge theory with adjoint scalar field that describes an infinite D1-string piercing the D3-brane, which we shall call the *fluxon*. Then we find its generalization which describes N D1-strings which pierce the D3-brane at various points. This solution will be called the N-*fluxon*.

Despite being infinite these string-like solitons are not translationally invariant — they depend on the specific point of intersection — although the equations of motion are translationally invariant. Thus the solitons we find are not translationally invariant, although the theory is. Thus the spectrum of small fluctuations should contain translational zero modes. However, we find that in the noncommutative directions these modes are essentially gauge transformations. Indeed, we show that in noncommutative gauge theory translations are equivalent to (large) gauge transformations plus shifts of the gauge field by a constant amount.

The fluxon solution is so simple that we are able to evaluate explicitly the complete spectrum of fluctuations about the soliton. This analysis is presented in section 4. We find that the fluctuating modes are those of fundamental strings. They fall into three classes. These correspond to light modes of fundamental strings attached to the D1-string and to ordinary gauge, scalar and fermion particles that can thought of as the light modes of fundamental strings attached to the D3-brane. In addition we find a set of modes with *discrete* spectrum of energies that correspond to fundamental strings that run between the D1-string and the D3-brane and are localized near the point of intersection.

In section 5 we study the dynamics of the modes of the fluxon — their propagation and their interaction with the perturbative gauge particles and with the localized string states. In section 6 we briefly generalize the discussion of fluctuations and interactions to the case of the N-fluxon.

Finally, in section 7, we show that, having constructed monopole-strings, we can also easily construct dyonic-strings. These have a natural interpretation as the reflection in the gauge theory of $(p,1)$ strings attached to the D3-brane. Similarly, we construct (p,q) fluxons, infinite (p,q)-strings piercing the D3-brane. We match the tension of the gauge theory strings with that of (p,q)-strings.

Section 8 contains some concluding remarks.

2. Noncommutative gauge theory

2.1 Notations and setup

Let us briefly review the framework of noncommutative gauge theory and establish notation. Consider space-time with coordinates x^i, $i = 1, \ldots, d$ which obey the following commutation relations:

$$[x^i, x^j] = i\theta^{ij}\,, \tag{2.1}$$

where θ^{ij} is a constant asymmetric matrix of rank $2r \leq d$. By noncommutative space-time we mean the algebra $\mathcal{A}_\theta$ generated by the x^i satisfying (2.1) , together with some extra conditions on the allowed expressions of the x^i. The elements of $\mathcal{A}_\theta$ can be identified with ordinary functions on $\mathbb{R}^d$, with the product of two functions f and g given by the Moyal formula (or star product):

$$f \star g\,(x) = \exp\left[\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x_1^i}\frac{\partial}{\partial x_2^j}\right] f(x_1)g(x_2)|_{x_1=x_2=x}\,. \tag{2.2}$$

For plane waves:

$$e^{i\vec{p}_1\cdot\vec{x}} \star e^{i\vec{p}_2\cdot\vec{x}} = e^{-\frac{i}{2}\vec{p}_1\times\vec{p}_2}\, e^{i(\vec{p}_1+\vec{p}_2)\cdot\vec{x}}\,, \tag{2.3}$$

where

$$\vec{p}_1 \times \vec{p}_2 = \theta^{ij} p_{1i} p_{2j} = -\vec{p}_2 \times \vec{p}_1\,. \tag{2.4}$$

We shall restrict our attention to the case of $\mathcal{N} = 4$ four dimensional U(1) super-Yang-Mills theory on a noncommutative space-time, with the noncommutativity parameter $\theta^{\mu\nu}$ being space-like. One can then choose coordinates so that

$$[x^1, x^2] = -i\theta\,, \qquad [x^3, \cdot\,] = [t, \cdot\,] = 0\,. \tag{2.5}$$

The lagrangian of a field theory involves derivatives. The derivative ∂_i is the infinitesimal automorphism of the algebra (2.1):

$$x^i \to x^i + \varepsilon^i\,, \tag{2.6}$$

where ε^i is a c-number. For the algebra (2.1) this automorphism is internal:

$$\partial_i \Psi = i\theta_{ij}[\Psi, x^j]\,, \tag{2.7}$$

where θ_{ij} is the inverse of θ^{ij}, namely $\theta_{ij}\theta^{jk} = \delta_i^k$. It is convenient to introduce the operators:

$$c = \frac{1}{\sqrt{2\theta}}\left(x^1 - ix^2\right), \qquad c^\dagger = \frac{1}{\sqrt{2\theta}}\left(x^1 + ix^2\right), \tag{2.8}$$

which obey:

$$[c, c^\dagger] = 1 .$$

Note that

$$\frac{\partial}{\partial x_1} = \frac{1}{\sqrt{2\theta}}[c - c^\dagger, \cdot\,], \qquad \frac{\partial}{\partial x_2} = \frac{i}{\sqrt{2\theta}}[c + c^\dagger, \cdot\,]. \tag{2.9}$$

Since $c, c^\dagger$ satisfy the commutation relations of the annihilation and creation operators we can identify functions $f(x_1, x_2)$ with functions of the $c, c^\dagger$ valued in the operators acting in the standard Fock space $\mathcal{H}$ of the creation and annihilation operators:

$$\begin{aligned}
\mathcal{H} &= \bigoplus_{n=0}^{\infty} \mathbf{C}|n\rangle \\
c^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\
c|n\rangle &= \sqrt{n}|n-1\rangle \\
\hat{n} &= c^\dagger c\,, \qquad \hat{n}|n\rangle = n|n\rangle \qquad \langle m|n\rangle = \delta_{mn}\,.
\end{aligned} \tag{2.10}$$

Since we shall be dealing with a scale invariant theories in which the only scale is θ we shall set $2\theta = 1$. When desired, θ can be introduced back simply by rescaling the coordinates, $x_i \to x_i/\sqrt{2\theta}$, for $i = 1, 2$.

The procedure that maps ordinary commutative functions onto operators in the Fock space is called Weyl ordering and is defined by:

$$\begin{aligned}
f(x) = f\left(z = x^1 - ix^2, \bar{z} = x^1 + ix^2\right) &\mapsto \\
\mapsto \hat{f}(c, c^\dagger) = \int f(x) \frac{\mathrm{d}^2x\mathrm{d}^2p}{(2\pi)^2}\, e^{i\left[\bar{p}_a(c-z) + p_a\left(c^\dagger{}_a - \bar{z}\right)\right]}\,.
\end{aligned} \tag{2.11}$$

It is easy to see that

$$\text{if } f \mapsto \hat{f}\,,\; g \mapsto \hat{g} \qquad \text{then } f \star g \mapsto \hat{f}\hat{g}\,. \tag{2.12}$$

A useful formula is for the matrix elements of $\hat{f}$ in the coherent state basis

$$\langle \xi|\hat{f}|\eta\rangle = \int f(z, \bar{z}) \frac{dz\, d\bar{z}}{(2\pi i)^2}\, e^{\xi\cdot\eta - 2(\xi - \bar{z})\cdot(\eta - z)}\,, \tag{2.13}$$

where $\langle\xi|$ and $|\eta\rangle$ are coherent states: $\langle\xi| = \langle\mathbf{0}|\exp(\xi c^\dagger)$, $|\eta\rangle = \exp(\eta c)|\mathbf{0}\rangle$. Translations in the Hilbert space are generated by $\hat{\partial}_i$, where

$$\hat{\partial}_1 = (c - c^\dagger) = -2ix_2\,, \qquad \hat{\partial}_2 = i(c + c^\dagger) = 2ix_1\,. \tag{2.14}$$

Thus if $f(x) \mapsto \hat{f}$, then $f(x + a) \mapsto \exp(a \cdot \hat{\partial})\hat{f}\exp(-a \cdot \hat{\partial})$.

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2.2 Gauge theory

The covariant derivative of a U(1) gauge field is then represented as the operator:

$$
\begin{aligned}
D_0 &= \partial_0 + A_0\,, & D_3 &= \partial_3 + A_3\,,\\
D &= \frac{1}{2}\left(D_1 + iD_2\right) = -c^\dagger + A\,, & \bar{D} &= \frac{1}{2}\left(D_1 - iD_2\right) = c + \bar{A}\,,
\end{aligned}
\tag{2.15}
$$

where A_μ are the anti-hermitean components of the gauge field and

$$
A = \frac{1}{2}\left(A_1 + iA_2\right)\,, \qquad \bar{A} = -A^\dagger = \frac{1}{2}\left(A_1 - iA_2\right).
$$

Under a gauge transformation

$$
D \to UDU^\dagger\,, \quad \bar{D} \to U\bar{D}U^\dagger\,; \qquad U^\dagger U = UU^\dagger = 1\,.
$$

The anti-hermitean field strength is $F_{\mu\nu} = [D_\mu, D_\nu] - i\theta_{\mu\nu}$.

The action for the $\mathcal{N} = 4$ supersymmetric noncommutative U(1) gauge theory is given by $(a = 1, \ldots, 6)$:

$$
\mathcal{L}(A) = -\frac{2\pi\theta}{g^2}\int \mathrm{d}t\mathrm{d}x_3\ \mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi_a D^\mu\Phi_a + \frac{1}{4}[\Phi_a, \Phi_b]^2\right] + \text{fermions}\,,
\tag{2.16}
$$

where the trace Tr over the Fock space states is equivalent to integration over the noncommuting coordinates x_1 and x_2.

2.3 Topological charges in noncommutative gauge theory

Just as in an ordinary gauge theory, noncommutative gauge theory has topological charges, e.g. magnetic fluxes or instanton numbers. These can be defined via integrals of the characteristic forms:

$$
\int \mathrm{Tr}\, e^{F/2\pi i}\,,
\tag{2.17}
$$

where now the integration over the noncommutative space-time $\int$ can be included in the definition of the trace Tr, with all the factors $2\pi\theta$ understood. In the commutative case there is an alternative definition, which involves patching the space-time with open domains, glueing functions etc. In the noncommutative case such a definition is lacking simply because one has to work (in some sense) globally over the noncommutative part of the space-time. However, for noncommutative $\mathbb{R}^2, \mathbb{R}^4$ there are "asymptotic" techniques for calculations of the topological charge.

Let us discuss noncommutative $\mathbb{R}^2$ for simplicity. Recall that we view the components A_1, A_2 of the gauge field on the noncommutative $\mathbb{R}^2$ as operators in the Fock space $\mathcal{H}$. Suppose we are looking for gauge field configurations with finite energy density when integrated over $\mathrm{d}x^1\mathrm{d}x^2$. Then, as $x_1^2 + x_2^2 \to \infty$ (in other words, when looking at the matrix elements of the operators between the states of high occupation numbers) the gauge fields must approach a pure gauge:

$$A_\mu \to U^\dagger \partial_\mu U \,, \tag{2.18}$$

where $U \in \mathrm{U}(\mathcal{H})$ is a unitary operator in $\mathcal{H}$. More precisely, since we understand the limit in (2.18) to mean:

$$\langle n|A_\mu|\bar{n}\rangle \to \langle n|U^\dagger \partial_\mu U|\bar{n}\rangle \,, \qquad \text{as } n, \bar{n} \to \infty \,, \tag{2.19}$$

we only require that U is well-defined and unitary on the subspace of $\mathcal{H}$ which contains the states $|n\rangle, |\bar{n}\rangle$ with sufficiently high $n, \bar{n}$. We can then continue U on the whole of $\mathcal{H}$ but it will, generically cease to be unitary on the whole of $\mathcal{H}$. The measure of the non-unitarity of U is its index:

$$\mathrm{Ind}\, U = \dim \mathrm{Ker}\, U - \dim \mathrm{Ker}\, U^\dagger \,.$$

If U can be deformed to the unitary operator then certainly $\mathrm{Ind}U = 0$. If, on the other hand, $\mathrm{Ind}U \neq 0$ then U cannot be deformed into a unitary operator. We could just as well take the difference of the dimensions of the kernels of the hermitean operators to define the unitarity obstruction:

$$\mathcal{I}_U = \dim \mathrm{Ker}\, U^\dagger U - \dim \mathrm{Ker}\, U U^\dagger \,.$$

The solutions which we shall discuss below will have $\mathcal{I}_U \neq 0$.

3. Monopoles and strings

3.1 BPS solitons

In our previous paper, [11] we discussed the noncommutative generalization of Nahm's equations that describe BPS solitons of gauge theory. We considered classical, static, solutions of the theory given by (2.16). To construct these solutions we set $A_0 = 0$ and chose $\Phi_a = \delta_{a1}\Phi$. Of course, given any particular solution for Φ we can write the general solution as $\Phi_a = \hat{\mathbf{n}}\Phi$, where $\hat{\mathbf{n}}$ lies on S^5. For this choice the BPS equations, that minimize the energy, are

$$[D_i, \Phi] = \pm B_i \,, \qquad B_i = \frac{1}{2}\epsilon_{ijk} F_{jk} \,. \tag{3.1}$$

We were able to find explicit solutions of these equations, by using the noncommutative Nahm equations. The solutions were given by the following expressions:

$$\Phi = \Phi(\hat{n}) = \frac{\zeta(\hat{n})}{\zeta(\hat{n}-1)} - \frac{\zeta(\hat{n}+1)}{\zeta(\hat{n})} \,, \tag{3.2}$$

$$A_3 = 0; \qquad A = c^\dagger \left(1 - \frac{\xi(\hat{n})}{\xi(\hat{n}+1)}\right) \,, \tag{3.3}$$

where the functions $\xi(\hat{n})$, $\zeta(\hat{n})$ are given in [11]. We shall not need here the explicit form of these functions.

The above soliton looks very simple far away from the origin, where it reduces to fields that describe a semi-infinite string along the positive x_3 axis plus a magnetic monopole at the origin. In particular,

$$\Phi \to -2x_3 P_0; \qquad B_3 \to 2P_0\,; \qquad \text{for large } x_3 > 0\,,$$

where $P_0 = |0\rangle\langle 0|$ is the projection operator in the Fock space $\mathcal{H}$ which projects onto the vacuum state. When translated back to ordinary position space the operator P_0 becomes $\exp[-\frac{x_1^2+x_2^2}{\theta}]$. Thus this soliton looks like a magnetic monopole at the origin that is attached to a semi-infinite string along the positive x_3 axis. In [11] we argued that this soliton corresponds precisely to a D1-string attached to a D3-brane at the origin. We calculated the tension (=energy per unit length) of the string and found that it is given by

$$T = \frac{2\pi}{g^2\theta}\,, \tag{3.4}$$

in complete agreement with the tension of a D1-string in the bulk, correctly scaled in the Seiberg-Witten decoupling limit. The D1-string, tilted by the background B-field, forms an angle ψ, $\tan\psi = 2\pi\alpha'/\theta$ with the D3-brane. Although this angle goes to zero in geometrical units in the Seiberg-Witten $\alpha' \to 0$ limit, in the gauge theory this angle can be observed by the asymptotic slope of the Higgs field, since α' is hidden when we replace the coordinates tranverse to the D3-brane (which have dimensions of length) by the scalar fields on the brane (with dimensions of mass).

3.2 An infinite string

The solution (3.2) and (3.3), discussed above describes a semi-infinite string, attached to a monopole — a reflection of a D1-string attached to a D3-brane. There are other BPS string configurations, whose decoupling limit in the presence of a background B-field might also give rise to noncommutative gauge theory solitons. We might consider an infinite D1-string that pierces the D3-brane. This is certainly BPS. The semi-infinite D1-string attached to the D3-brane can be regarded as the limit of a D1-string stretched between two D3-branes, in the limit where the separation between the D3-branes goes to infinity. Before taking the limit, this corresponds, to a monopole in the U(2) gauge theory, that describes the low energy dynamics of the D3-branes, broken to U(1) by separating the branes. The distance between the branes is proportional to the vacuum expectation value of the Higgs field and, when this goes to infinity, only the U(1) gauge degrees of freedom, and the massless Higgs field survive.

To obtain the piercing string we could start with 3 D3-branes and break the SU(3) gauge theory to U(1) $\times$ U(1) $\times$ U(1) by separating all the branes. We can then stretch one D1-string from the middle D3-brane up (in the Φ direction) and another D1-string down. In the limit of infinite separation, and in the presence of a background B-field, this will describe two semi-infinite strings, in the $+x_3$ and in the $-x_3$ directions. If we bring their origins together at the same point on the D3-brane we will obtain an infinite, piercing string — the fluxon. The monopole and the and anti-monopole at the intersectiuon point will annihilate and we should be left only with the flux tube.

We can obtain the piercing string solution by manipulating the solution described in [11]. Consider the Higgs field $\Phi(x_3)$. Far away from the origin it approaches $-2x_3P_0$, up to terms that fall off as $1/x_3$. If we consider a solution translated in the x_3 direction by an amount δ, in the limit as $\delta \to \infty$, then for any finite x_3, $\Phi = -2(x_3+\delta)P_0 + O(1/\delta)$. It is easy to verify that the components $A, \bar{A}$ are given, in this limit by,

$$A = c^\dagger \left(1 - \sqrt{\frac{\hat{n}}{\hat{n}+1}}\right), \qquad \bar{A} = \left(\sqrt{\frac{\hat{n}}{\hat{n}+1}} - 1\right) c\,, \tag{3.5}$$

where $\hat{n}$ is the number operator: $\hat{n} = c^\dagger c$. Equivalently,

$$A = \sum_{n=0}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right) |n+1\rangle\langle n|\,. \tag{3.6}$$

Finally, we can drop the x_0P_0 term in Φ, since the equations of motion for Φ only involve derivatives with respect to x_3. Thus, the above gauge fields and

$$\Phi = -2x_3P_0\,, \tag{3.7}$$

yield a BPS solution of the static gauge theory.

It is easy to check that this is indeed a BPS soliton. For the above configuration, $A_3 = 0$ and $A_1$and A_2 are independent of x_3. Thus it follows that the only nonvanishing component of the magnetic field is B_3 since $B_i \propto \epsilon_{ij}\partial_3 A_j$, $4i, j = 1, 2$, and

$$B_3 = 2\left([\bar{D}, D] + 1\right), \tag{3.8}$$

where

$$D = -c^\dagger\sqrt{\frac{\hat{n}}{\hat{n}+1}}, \quad \bar{D} = \sqrt{\frac{\hat{n}}{\hat{n}+1}}c \quad \Rightarrow \quad -\bar{D}D = \hat{n}\,, \quad -D\bar{D} = \hat{n} - 1 + P_0\,. \tag{3.9}$$

This is consistent with the BPS equations since $\Phi \propto P_0$ and thus

$$\hat{n}P_0 = P_0\hat{n} = cP_0 = P_0c^\dagger = 0 \Rightarrow [D, \Phi] = [\bar{D}, \Phi] = 0\,. \tag{3.10}$$

It is instructive to check the equations (3.10) in the coordinate space, for example:

$$c\,P_0 \mapsto (x_1 - ix_2) \star \exp\left(-2\left(x_1^2 + x_2^2\right)\right) = 0\,.$$

Finally, we have

$$B_3 = 2P_0 = -\partial_3\Phi\,. \tag{3.11}$$

This solution clearly describes an infinite flux tube of magnetic field along the x_3-axis, localized in the noncommutative plane, with a linear Φ field along the tube that corresponds to the extension of the tube into a direction transverse to the D3-brane. Consequently, we shall call this soliton a *fluxon.*[1]

The fluxon solution for the gauge field is almost a pure gauge. We can write the covariant derivatives, $\bar{D}$ and D, as

$$\bar{D} = ScS^\dagger\,, \qquad D = -Sc^\dagger S^\dagger\,, \tag{3.12}$$

where

$$\begin{aligned} S &= c^\dagger \frac{1}{\sqrt{\hat{n}+1}} = \sum_{n=0}^{\infty} |n+1\rangle\langle n|\,, \quad S|n\rangle = |n+1\rangle\,, \\ S^\dagger &= \frac{1}{\sqrt{\hat{n}+1}} c = \sum_{n=0}^{\infty} |n\rangle\langle n+1|\,, \quad S^\dagger|n\rangle = |n-1\rangle, \quad n>0\,, \quad S^\dagger|0\rangle = 0\,. \end{aligned} \tag{3.13}$$

If S was unitary this would be a pure gauge, but

$$S^\dagger S = 1\,, \quad SS^\dagger = 1 - P_0\,. \tag{3.14}$$

Thus S is unitary in the subspace picked out by $1-P_0$, and only fails to be unitary in the vacuum state. In terms of the indices discusses above, the index of $S^\dagger$ or of $SS^\dagger$ is:

$$\mathrm{Ind}\, S^\dagger = \mathcal{I}_{S^\dagger} = 1\,.$$

It is also interesting to consider the commutative limit $\theta \to 0$ of the fluxon solution. The vacuum projector corresponds to the gaussian packet in the noncommutative plane:

$$P_0 \to 2\exp\left[-\frac{x_1^2+x_2^2}{\theta}\right] \to 2\pi\theta\delta(x_1)\delta(x_2) \qquad \text{as } \theta \to 0\,.$$

Thus in this limit the fields are given, in coordinate space by:

$$\Phi = -2\pi\delta(x_1)\delta(x_2)x_3\,, \qquad A = i\frac{x_1\,dx_2 - x_2\,dx_1}{x_1^2+x_2^2}\,, \tag{3.15}$$

and the magnetic field is

$$B = dA = 2\pi i\delta(x_1)\delta(x_2)\,, \tag{3.16}$$

which clearly satisfies:

$$dA + i \star d\Phi = 0\,. \tag{3.17}$$

Thus in the commutative theory we have a singular solenoid extending along the x_3 axis. The fluxon has a magnetic charge, defined as:

$$Q_m = \frac{1}{2\pi i}\int F_{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 = -i\theta\,\mathrm{Tr}\,F_{12} = +1\,. \tag{3.18}$$

[1] A similar solution was constructed in [12, eq. (31)].

3.3 Higher charge fluxons

It turns out that a very simple modification of (3.13) produces flux tubes of higher magnetic charge. The idea is to use the fact that a subspace of the Fock space $\mathcal{H}$, orthogonal to a finite-dimensional subspace, is isomorphic to $\mathcal{H}$ as a Hilbert space. Let S_N be a unitary isomorphism

$$S_N : \mathcal{H} \to \mathcal{H}_N \,, \qquad \dim(\mathcal{H}/\mathcal{H}_N) = N \,. \tag{3.19}$$

We assume that

$$\mathcal{H} = V_N \oplus \mathcal{H}_N \,, \qquad \dim V_N = N \,, \qquad V_N \perp \mathcal{H}_N \,, \tag{3.20}$$

where $\perp$ means orthogonality in the sense of the $\langle | \rangle$ hermitean inner product on $\mathcal{H}$. We have:

$$S_N^\dagger S_N = 1 \,, \qquad S_N S_N^\dagger = 1 - P_N \,, \tag{3.21}$$

where P_N is the orthogonal projection onto V_N.

Then the N-fluxon is given by:

$$\begin{aligned} D &= -S_N \, c^\dagger \, S_N^\dagger \,, \qquad \bar{D} = S_N \, c \, S_N^\dagger \,, \\ \Phi &= -2x_3 P_N \,. \end{aligned} \tag{3.22}$$

By a unitary gauge transformation we can always bring S_N to the following form:

$$S_N = S^N \,, \qquad S_N |n\rangle = |n+N\rangle \,, \qquad P_N = \sum_{m=0}^{N-1} |m\rangle\langle m| \,. \tag{3.23}$$

The magnetic charge (per unit length) of the N-fluxon is clearly N:

$$Q_m = \frac{1}{2\pi i} \int F_{12} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = N \,, \tag{3.24}$$

which also equals the index of $S_N^\dagger$:

$$\mathrm{Ind}\, S_N^\dagger = \mathcal{I}_{S_N^\dagger} = 1 \,.$$

The N-fluxon exhibited above breaks the $\mathrm{U}(\infty)$ (more precisely, $\mathrm{U}(\mathcal{H})$) gauge symmetry

$$D \to UDU^\dagger \,, \quad \bar{D} \to U\bar{D}U^\dagger \,, \quad \Phi \to U\Phi U^\dagger , \quad U^\dagger U = UU^\dagger = 1 \,, \quad U \in \mathrm{U}(\mathcal{H}) \,. \tag{3.25}$$

down to $\mathrm{U}(N)$, where the $\mathrm{U}(N)$ acts within V_N. However, we can break this symmetry further, all the way down to $\mathrm{U}(1) \times \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$, and separate the N-fluxons by shifting:

$$\Phi \to -2x_3 P_N + P_N D_N P_N \,, \tag{3.26}$$

where D_N is a constant diagonal $N \times N$ matrix, with eigenvalues d_i, $i = 1, \ldots, N$. This will clearly solve the BPS equations and represent N-fluxons intersecting the D3-brane at positions $x_3 = d_i/2$. If all the d_i are different then $\mathrm{U}(\infty)$ will be broken down to $\mathrm{U}(1)^N$.

Also notice that, as expected, the $\mathcal{N} = 4$ theory has solutions describing D1-strings separated in all six directions:

$$\mathbf{\Phi} = -2x_3 \hat{\mathbf{n}} P_N + P_N \mathbf{D}_N P_N \,, \tag{3.27}$$

where $\mathbf{D}_N$ is a sextet (transforming in the representation $\mathbf{6}$ of the SO(6) R-symmetry group) of diagonal $N \times N$ matrices.[2]

3.4 Properties of fluxons

The energy density of the N-fluxon is

$$\mathcal{E} = \frac{1}{2g^2}\left(\vec{B}^2 + 4[D,\Phi][\bar{D},\Phi] + (\partial_3\Phi)^2\right) = \frac{4}{g^2} P_N \,, \tag{3.28}$$

and thus the gauge invariant energy per unit (x_3) length, the tension, is (restoring θ)

$$T = \frac{2\pi N}{g^2\theta} \,. \tag{3.29}$$

The 1-fluxon is not translationally invariant. It corresponds to a D1-string that pierces the D3-brane at a specific point. Of course we can obtain other solutions by translating $x \to x + a$. This is obvious in the case of translations along the x_3 axis and $\partial_3 \Phi = -2P_0$ will be a normalizable (per unit length) zero mode of the soliton ($\partial_3 A = 0$). Similarly, in the case of the separated N-fluxon solution (3.26) there are N translational zero modes that shift the d_i.

Translations of the noncommutative coordinates are subtler. Translations of operators in the noncommutative directions are generated by the operators $\hat{\partial}$, defined in (2.14). Thus we can translate our solution by an amount $a = (a_1, a_2)$ by performing

$$\Phi \to \exp(a \cdot \hat{\partial})\Phi \exp(-a \cdot \hat{\partial}) \,, \qquad A \to \exp(a \cdot \hat{\partial}) A \exp(-a \cdot \hat{\partial}) \,.$$

This is a gauge transformation of the Higgs field, $\Phi \to U(a)\Phi U^\dagger(a)$, where

$$U(a) = \exp(-a \cdot \hat{\partial}) = \exp(i a_i \theta^{-1}_{ij} \hat{x}_j) \,. \tag{3.30}$$

Acting on the gauge field this transformation yields

$$A \to U(a) A U^\dagger(a) = \left[U(a)(A - c^\dagger)U^\dagger(a) + c^\dagger\right] + U(a)[c^\dagger, U^\dagger(a)] \equiv \delta_1 A + \delta_2 A \,. \tag{3.31}$$

[2] The complete solution, which depends on $8N$ moduli, together with the proof of its uniqueness, will be presented elsewhere [13].

The first term, $\delta_1 A$, in (3.31) is a gauge transformation and the second term, $\delta_2 A$, is a constant, c-number, shift of the gauge field.

$$\delta_2 A = U(a)[c^\dagger, U^\dagger(a)] = -(a^1 + ia^2)\,. \tag{3.32}$$

Both of these, gauge transformations and constant shifts of the gauge field, are symmetries of the action. What is unusual about noncommutative gauge theories is that *translations in the noncommutative directions are equivalent to a combination of a gauge transformation and a constant shift of the gauge field.* This explains why in noncommutative gauge theories there do not exist local gauge invariant observables, since by a gauge transformation we can effect a spatial translation! This is analogous to the situation in general relativity, where translations are also equivalent to gauge transformations (general coordinate transformations) and one cannot construct local gauge invariant observables. The fact that spatial translations are equivalent to gauge transformations (up to global symmetry transformations) is one of the most interesting features of noncommutative gauge theories. These theories are thus toy models of general relativity — the only other theory that shares this property.

The gauge transformation that corresponds to a translation is a *large gauge transformation* that does not approach the identity at spatial infinity; namely $U \sim 1 + i\lambda$, where $\lambda = -ia^k \theta^{-1}_{kl} x^l$. Thus these gauge transformations are analogous to the large gauge transformations that take us from one winding number vacuum to another in non-abelian gauge theory.

When the string is quantized we will have to introduce collective coordinates for these zero modes and construct wave packets with definite momenta, much as we construct ϑ-vacua in non-abelian gauge theory or states of definite momentum in general relativity.

4. Fluctuations of the string(s)

We shall now discuss the fluctuations of the theory about the fluxon solution. We expect to find a variety of fluctuations when we linearize the theory about the fluxon. First, we certainly expect to find a continuous spectrum of photons and scalars (plus fermionic partners), which look like the modes of the theory in the vacuum, far away from the soliton. From the point of view of the string theory, these modes correspond to light fundamental strings attached to the D3-brane. We will therefore refer to them as 3-3 modes.

In addition we might expect to find the fluctuations of the string itself. Since, we argue, the noncommutative gauge theory soliton represents a D1-string, we might expect to find the modes of a supersymmetric D1-string propagating in 10 dimensions. The N-fluxon is a gauge theory soliton that represents N D1-strings. The fluctuations of these strings, whose tension is of order $1/g^2\theta$, cannot be seen in a

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small g^2 expansion about the string. In other words, the massive D1-string states have energies of order $1/g^2$, and to see these we would require nonperturbative techniques. However, there are other modes of the D1-strings that are visable in the semiclassical domain of the gauge field theory — namely the modes that arise from fundamental open strings attached to the D1 strings. In the $\alpha' \to 0$ limit we are taking these modes should be described by a supersymmetric U(N) Yang-Mills theory living on the 2 dimensional world sheet of the D1-strings. If all the fluxons are at the same point the U(N) gauge symmetry should be unbroken. If they are separated, as in (3.26) and (3.27), than the gauge theory will be in the broken symmetry Higgs phase. We indeed find such a spectrum of small fluctuations. We refer to these as 1-1 modes.

Finally, there are some extra stringy modes. Given a D1-string that is attached to a D3-brane, one can attach a fundamental string to the D1 at one end and to the D3-brane at the other. These too should be reflected in the gauge theory in the presence of our D1 soliton. In the decoupling limit we are considering, the only modes of these fundamental strings that could survive would have to be short and thus localized around the point ($\vec{x} = 0$) where the D-branes intersect. Indeed, we find such modes, localized about the origin, with discrete energies. We shall refer to these as 1-3 modes.

Let us restrict our attention to the $N = 1$ fluxon. We shall discuss the generalization to the case of the N fluxon later.

4.1 Expansion of the action

It is convenient to view the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory as a dimensionally reduced ten dimensional SYM theory. We shall use the indices $\mu, \nu, \ldots = 0, 1, 2, 3$ to denote four dimensional quantities, the indices $i, j, \ldots = 1, 2, 3$ to denote three dimensional quantities, and the indices $A, B, \ldots = 0, 1, \ldots, 9$ to denote ten dimensional ones. The scalar fields can be regarded as the extra six components of the ten-dimensional gauge field,

$$\Phi_a = iA_{3+a}\,, \qquad a = 1\,, \ldots 6\,, \qquad D_B = [A_B, \cdot\,]\,, \qquad B = 4, \ldots, 9\,. \tag{4.1}$$

They are hermitean since our gauge fields are anti-hermitean. The lagrangian can then be written as

$$\mathcal{L} = -\frac{1}{4g^2}\left([D_A, D_B]^2 + \bar{\lambda}_{\dot{\alpha}}\Gamma_B^{\dot{\alpha}\,\alpha}[D_B, \lambda_\alpha]\right). \tag{4.2}$$

Here we have included the fermionic partners, λ_α, of the N=4 supermultiplet. We expand the SYM action about the soliton, A_B^0, for which the only nonvanishing components are for $B = 1, 2, 3, 4$. Expanding

$$A_B = A_B^0 + g a_B\,, \tag{4.3}$$

and fixing the gauge by imposing the condition

$$[D^0_B, a_B] = 0 \tag{4.4}$$

on the fluctuations, we obtain:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \\
\mathcal{L}_0 &= \frac{2\pi}{g^2\theta} P_0 \\
\mathcal{L}_2 &= -\frac{1}{2}\left((D^0_B a_C)^2 + 2F^0_{BC}[a_B, a_C] + \bar{\lambda}\mathcal{D}^0\lambda \right) \\
\mathcal{L}_3 &= -g\left(D^0_B a_C [a_B, a_C] + \bar{\lambda}\Gamma_A [a_A, \lambda] \right) \\
\mathcal{L}_4 &= -\frac{g^2}{4}[a_A, a_B]^2 .
\end{aligned} \tag{4.5}$$

The only nonvanishing components of the field strength in the background given by (3.5), (3.7) are:

$$F^0_{12} = -2iP_0\,, \qquad F^0_{34} = 2iP_0\,. \tag{4.6}$$

The only nonvanishing components of the covariant derivative are D^0_A, A=$0, 1, 2, 3, 4$.

4.2 The linearized action

To linear order (g^0) we have simply a bunch of free gauge fields, scalars and fermions. We wish to diagonalize the quadratic form $-\frac{1}{2}a_A\Delta a_A + \bar{\lambda}_{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}\lambda_\alpha$ that appears in S_2. The bosonic operator Δ, corresponding to S_2, is:

$$-\Delta a_C \equiv -[D^0_B, [D^0_B, a_C]] + 2[F^0_{CB}, a_B]\,. \tag{4.7}$$

Each component of the gauge field a_B is an (t, x^3)-dependent operator in the Fock space $\mathcal{H}$. The single fluxon corresponds to the splitting of the Fock space into the sum of a one-dimensional subspace $V_1 = \mathbf{C}|0\rangle$ and the orthogonal complement $\mathcal{H}_1$:

$$\mathcal{H} = V_1 \oplus \mathcal{H}_1\,.$$

Accordingly, the space End$(\mathcal{H})$ of the operators in the Fock space splits as a direct sum of subspaces:

$$\mathrm{End}\mathcal{H} = V_{11} \oplus V_{13} \oplus V_{31} \oplus V_{33}\,, \tag{4.8}$$

where

$$V_{11} = V_1 \otimes V_1\,, \qquad V_{13} = V_1 \otimes \mathcal{H}_1\,, \qquad V_{31} = \mathcal{H}_1 \otimes V_1\,, \qquad V_{33} = \mathcal{H}_1 \hat{\otimes} \mathcal{H}_1\,. \tag{4.9}$$

The operator Δ preserves each of these subspaces. We shall see that these subspaces contain the 1-1, 1-3, (their conjugates 3-1) and 3-3 modes respectively, hence their names.

Moreover, one can also split the space $\mathbb{R}^{10} = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^{1,5}$ of the Lorentz indices A, as follows:

$$(1,2) \oplus (3,4) \oplus (0,5,\ldots,9)\,.$$

For the $1,2$ subspace we introduce the components of the gauge field $X = \frac{1}{2}(a_1 + ia_2)$, for which

$$-\Delta X = -[D^0_A[D^0_A\,, X]] + 4[P_0, X]\,. \tag{4.10}$$

For the $3,4$ subspace we introduce the combination $Y = \frac{1}{2}(a_3 + ia_4)$, for which

$$-\Delta Y = -[D^0_A[D^0_A\,, Y]] - 4[P_0, Y]\,. \tag{4.11}$$

For the other $B = 5,\ldots,9$ components of the gauge field the operator Δ acts in a simpler fashion:

$$-\Delta a_B = -[D^0_A[D^0_A\,, a_B]\,. \tag{4.12}$$

Actually, both $[P_0,\cdot]$ and $[D^0[D^0,\cdot]]$ preserve the decomposition (4.8), therefore the classification of the eigenstates according to their 11, 13, 33 types holds for any $A = 0,\ldots,9$.

Now let us discuss the operator

$$\begin{aligned} D^2\mathcal{O} \equiv \left[D^0_A, \left[D^0_A, \mathcal{O}\right]\right] &= -\partial_t^2\,\mathcal{O} + \partial_3^2\,\mathcal{O} + 2[D,[\bar{D},\mathcal{O}]] + 2[\bar{D},[D,\mathcal{O}]] - [\Phi,[\Phi,\mathcal{O}]] \\ &= -(\partial_t^2 - \partial_3^2)\mathcal{O} \,-\, 4x_3^2\,[P_0,[P_0,\mathcal{O}]] + \\ &\quad + 2\left[-c^\dagger\sqrt{\frac{\hat{n}}{\hat{n}+1}}, \left[\sqrt{\frac{\hat{n}}{\hat{n}+1}}c, \mathcal{O}\right]\right] + 2\left[\sqrt{\frac{\hat{n}}{\hat{n}+1}}c, \left[-c^\dagger\sqrt{\frac{\hat{n}}{\hat{n}+1}}, \mathcal{O}\right]\right]. \end{aligned} \tag{4.13}$$

When restricted to the V_{ab} subspaces in (4.8) it simplifies to:

$$\begin{aligned} V_{11}:&\ \ -D^2 = \partial_t^2 - \partial_3^2\,, \\ V_{13}, V_{31}:&\ \ -D^2 = \partial_t^2 - \partial_3^2 + 4x_3^2 + 2(2\hat{n}+1)\,, \\ V_{33}:&\ \ -D^2 = \partial_t^2 - \partial_3^2 - 2(\bar{\partial}_{\bar{z}} - z)(\partial_z - \bar{z}) - 2(\partial_z - \bar{z})(\bar{\partial}_{\bar{z}} - z)\,. \end{aligned} \tag{4.14}$$

We have introduced, in the above, the following conventions. For V_{13} we define: $\hat{n}|0\rangle\langle m| = m|0\rangle\langle m|$ and for V_{31}: $\hat{n}|m\rangle\langle 0| = m|m\rangle\langle 0|$. For V_{33} we introduce the following representation. For an operator $A \in V_{33}$ we define a function of two variables $z, \bar{z}$ as follows:

$$A(z,\bar{z}) = \sum_{k,l>0} \langle k|A|l\rangle \frac{z^{k-1}\,\bar{z}^{l-1}}{\sqrt{(k-1)!(l-1)!}}\,. \tag{4.15}$$

Then the operators $D, \bar{D}$ acting on $a \in V_{33}$ will be represented as the operators

$$D \sim \bar{\partial}_{\bar{z}} - z\,, \qquad \bar{D} \sim \partial_z - \bar{z} \tag{4.16}$$

acting on $A(z,\bar{z})$. The gauge condition (4.4) also preserves the decomposition (4.8). For the individual subspaces the gauge condition reads:

$$\begin{aligned} V_{11}: &\quad -\partial_t a_0 + \partial_3 a_3 = 0 \\ V_{13}, V_{31}: &\quad -\partial_t a_0 + \partial_3(Y - \bar{Y}) - [\Phi, Y + \bar{Y}] + 2[\bar{D}, X] - 2[D, \bar{X}] = 0 \\ V_{33}: &\quad -\partial_t a_0 + \partial_3 a_3 - 2(\bar{\partial}_{\bar{z}} - z)\bar{Y}(z,\bar{z}) + 2(\partial_z - \bar{z})Y(z,\bar{z}) = 0\,. \end{aligned} \tag{4.17}$$

4.3 The spectrum

Now we can easily diagonalize Δ. We start with the V_{33} subspace. In this space, for all values of the index A, the operator Δ coincides with D^2 since all components are orthogonal to $|0\rangle$ and thus they commute with P_0. As such it has the following eigenvectors:

$$a_A(t, x_3, z, \bar{z}) \sim \zeta_A(\omega, \vec{k}) e^{i(\omega t - k_3 x_3 - k_{\bar{z}} z - k_z \bar{z}) + z\bar{z}}\,, \tag{4.18}$$

with eigenvalues

$$\omega^2 - k_3^2 - 4k_z k_{\bar{z}} \equiv \omega^2 - \vec{k}^2\,. \tag{4.19}$$

The gauge condition (4.17) implies that the polarization vector $\zeta_A(\vec{k})$ must be transverse: $-\omega\zeta_t + k_3\zeta_3 + 2k_z\zeta_{\bar{z}} + 2k_{\bar{z}}\zeta_z = 0$. In sum, this branch of the spectrum describes photons, scalars (and as we shall see later, their superpartners) propagating along the three-brane worldvolume. As usual, for on-shell quanta, $\omega^2 = \vec{k}^2$ the gauge condition has a residual gauge freedom, which can be used to set $\zeta_t = 0$.

The V_{11} branch of the spectrum is also simple. All components are proportional to P_0 and thus commute with P_0. Consequently Δ coincides with $-\partial_t^2 + \partial_3^2$ for all A and its eigenvectors are:

$$a_A \sim \zeta_A(\omega, k) e^{i(\omega t - k x_3)}\,, \tag{4.20}$$

with eigenvalues $\omega^2 - k^2$ and the condition on the polarization is:

$$-\omega\zeta^0 + k\zeta^3 = 0\,.$$

Again, for the on-shell quanta, $\omega = \pm k$, and one can gauge both ζ^0 and ζ^3 away, leaving eight physical modes. These modes correspond to the ground states of the open fundamental strings attached to the D1-string. Indeed, these are the modes of a 1+1 dimensional $\mathcal{N} = 8$ supersymmetric U(1) gauge theory living on the world sheet of the D1-string.

Finally, we need to solve for the V_{13}, V_{31} branches of the spectrum. Let us look at V_{13} spanned by the operators $\mathcal{O}_m = |0\rangle\langle m|, \quad m > 0$. The discussion of V_{31} will be similar. The spectrum of the operator Δ depends on the index A. All eigenfunctions have the same form:

$$a_A(t, \vec{x}) \sim \zeta_A(\omega, n, m)\, H_n(x_3) e^{-x_3^2}\, \mathcal{O}_m\,. \tag{4.21}$$

Here $H_n(x_3)$ denotes the n'th normalized Hermite polynomial:

$$H_n(x) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n\, n!}} e^{-x_3^2} \partial_3^n e^{x_3^2} \,. \tag{4.22}$$

The eigenvalues of Δ depend on A. There are three cases:

- **1,2** the eigenvectors of Δ are:

  $$X(n,m) \sim H_n(x_3)\, e^{i\omega t - x_3^2} \mathcal{O}_m, \quad m > 0,\ n \geq 0\,,$$

  with eigenvalue:

  $$\omega^2 - 4(n+m+1)\,.$$

- **3,4** the eigenvectors of Δ are:

  $$Y(n,m) \sim H_n(x_3) e^{i\omega t - x_3^2} \mathcal{O}_m\,, \qquad m > 0\,,\ n \geq 0\,,$$

  with eigenvalue:

  $$\omega^2 - 4(n+m-1)\,.$$

- **5,6,7,8,9** the eigenvectors of Δ are:

  $$Z(n,m) \sim H_n(x_3)\, e^{i\omega t - x_3^2} \mathcal{O}_m\,,\ Z = a_A,\ A \neq 1,2,3,4\,, \qquad m > 0,\ n \geq 0\,,$$

  with eigenvalue:

  $$\omega^2 - 4(n+m)\,.$$

The gauge condition imposes a relation between a_0, X and Y. As for the remaining five components, a_A, $A > 4$, there is no constraint.

In this sector the spectrum is discrete and the states are localized about $x_3 = 0$. This is expected for the 1-3 modes, which we argued should come from fundamental strings, stretched between the D1-string and the D3-brane. Notice that the $m = 1, n = 0$ branch of the spectrum $3,4$ corresponds to a zero mode. We discuss its meaning below. The $5,\ldots,9$ and $1,2$ branches of the spectrum have a mass gap.

4.4 Normalization of the modes

We have three branches of the spectrum of the bosonic fluctuations around our solution. We have to normalize them properly.

The norm on the fluctuations comes from the natural metric on the gauge fields:

$$\|a_A\|^2 = -\int \mathrm{d}t\mathrm{d}x_3\ \mathrm{Tr}\ a_A^2\,. \tag{4.23}$$

With the gauge condition $D_B^0 a_B = 0$, the norm on the gauge fixed fluctuations is given by the same formula (4.23).

The generic fluctuation can be decomposed as follows:

$$\begin{aligned}
a_A(\vec{x},t) = &\int i\frac{d\omega dk}{(2\pi)^2}\,\zeta_A(\omega,k)e^{i(\omega t-kx_3)}|0\rangle\langle 0| + \\
&+\int i\frac{d\omega}{2\pi}\sum_{n\geq 0,m\geq 1}\zeta_A(\omega,n,m)H_n(x_3)e^{i\omega t-x_3^2}|0\rangle\langle m| + \\
&+\int i\frac{d\omega}{2\pi}\sum_{n\geq 0,m\geq 1}\bar{\zeta}_A(-\omega,n,m)H_n(x_3)e^{i\omega t-x_3^2}|m\rangle\langle 0| + \\
&+\int i\frac{d\omega d^3\vec{k}}{(2\pi)^4}\,\zeta_A(\omega,\vec{k})\,S\,e^{i(\omega t-\vec{k}\cdot\vec{x})}\,S^\dagger\,. \qquad (4.24)
\end{aligned}$$

The condition that a_A is anti-hermitean translates into:

$$\zeta_A(-\omega,-k) = \bar{\zeta}_A(\omega,k)\,, \qquad \zeta_A(\omega,\vec{k}) = \bar{\zeta}_A(-\omega,-\vec{k})\,. \qquad (4.25)$$

Now we substitute (4.24) into (4.23) and get:

$$\begin{aligned}
\|a_A\|^2 = \int\frac{d\omega}{2\pi}\Bigg(&\int\frac{dk}{2\pi}|\zeta_A(\omega,k)|^2 + \sum_{n\geq 0,m\geq 1}\bar{\zeta}_A(-\omega,n,m)\zeta_A(\omega,n,m) + \\
&+\int\frac{d^3\vec{k}}{(2\pi)^3}|\zeta_A(\omega,\vec{k})|^2\Bigg)\,, \qquad (4.26)
\end{aligned}$$

where we used the formulae:

$$\mathrm{Tr}\,SBS^\dagger = \mathrm{Tr}\,B\,, \qquad \mathrm{Tr}\,S^\dagger BS = \mathrm{Tr}\,B - \langle 0|B|0\rangle\,, \qquad \mathrm{Tr}\,e^{i(\kappa c^\dagger+\bar{\kappa}c)} = \delta^{(2)}(\kappa)\,. \quad (4.27)$$

Summarizing, the fluctuations are decomposed as in (4.24) and the eigenvalues of the various terms in (4.24) are given by:

$$\begin{aligned}
V_{11}:&\ \ \omega^2-k^2 \\
V_{13},V_{31}:&\ \ \omega^2-4(m+n+(-1,0,1)) \\
V_{33}:&\ \ \omega^2-\vec{k}^2 \qquad (4.28)
\end{aligned}$$

from which we can read off the propagators.

4.5 Moduli and translational modes

The spectrum of the fluctuations computed above contains three types of zero modes: in the 1-1, 1-3 and 3-3 sectors respectively. In the 1-1 sector we should find the translational zero modes of the soliton. The zero mode corresponding to translations in the x_3 direction is simply $\partial_3\Phi = -2P_0$, or more generally $\zeta^A(\omega=0,k=0)\,P_0$, eight for all eight transverse components of the string. The translational modes in the noncommutative directions, as discussed above, are infinitesimal gauge transformations.

Let us now look at the $1-3$ sector. Here we find a zero mode that we shall interpret as a moduli of the solution space of soliton strings. This zero mode is given by:

$$a_3 + \varphi = \eta|0\rangle\langle 1|\, e^{-x_3^2}\,, \tag{4.29}$$

where η is a complex number, from which we get:

$$\varphi = \frac{1}{2}\left(\eta|0\rangle\langle 1| + \bar{\eta}|1\rangle\langle 0|\right) e^{-x_3^2}\,. \tag{4.30}$$

To understand this mode it is instructive to look at the eigenvalues of the perturbed Higgs field:

$$\Phi_\epsilon = \Phi + \epsilon\varphi = -2x_3\,|0\rangle\langle 0| + \frac{\epsilon}{2}\left(\eta|0\rangle\langle 1| + \bar{\eta}|1\rangle\langle 0|\right) e^{-x_3^2}\,. \tag{4.31}$$

Let us redefine $\epsilon\eta/2 \to \eta$. The matrix (4.31) is a 2×2 matrix which is simple to diagonalize: the eigenvalues are the roots of the quadratic equation:

$$\lambda(\lambda + 2x_3) = |\eta|^2 e^{-2x_3^2}\,. \tag{4.32}$$

For $x_3 \to \pm\infty$ one of the eigenvalues goes to zero, while the other behaves like $-2x_3$. Since $|\eta|^2 e^{-2x_3^2} > 0 > -x_3^2$ the eigenvalues never cross. It means that the spectral surface associated with Φ contains two branches which look like two semi-infinite strings, separated by the distance of order $|\eta|$. They become visible if we rewrite (4.31) in position space as (we choose $\eta = 1$),

$$\Phi_\epsilon(x_1, x_2, x_3) = \exp[-2(x_1^2 + x_2^2)]\left[-2x_3 + 2\epsilon x_1 \exp(-x_3^2)\right]. \tag{4.33}$$

The interpretation of this zero mode is clear — it corresponds to splitting the fluxon into two semi- infinite strings of the type constructed in [11]. If we were to continue to separate these strings they would approach, for large separation in the x_1 direction, two semi-infinite strings as constructed in [11]. It is not trivial to construct the explicit solution for two semi-infinite strings separated by a finite distance, since we would have to relax the simplifying axially symmetric ansatz of [11].

5. Interactions

The interactions of the various modes in the presence of the soliton are described by the nonlinear terms in the lagrangian, $\mathcal{L}_3$ and $\mathcal{L}_4$.

The vertices of the gauge theory are organized in a manner which mimics the disc amplitudes of the open string theory with the D1 or D3 boundary conditions. The cubic vertices correspond to the three point correlation function, while the quartic vertices come from the four point function (the latter also contains the contribution of the tree diagram with two cubic vertices).

To proceed further we need to decompose the product $a_A a_B$:

$$a_A a_B = (a_A a_B)^{11}\,|0\rangle\langle 0| + (a_A a_B)^{13}_m\,|0\rangle\langle m| + (a_A a_B)^{31}_m\,|m\rangle\langle 0| + (a_A a_B)^{33}_{m\bar{m}}\,|m\rangle\langle \bar{m}|\,. \tag{5.1}$$

We shall omit summations over repeated indices or integrations over repeated momenta. We also omit the factor $e^{i(\omega+\omega')t}$ in front of everything, and we assume that frequency of the a_A mode is ω, while a_B has the frequency ω' (the frequencies are to be integrated over, of course). Finally, $\vec{k} = (k, \kappa, \bar{\kappa})$, $\vec{k}\cdot\vec{x} = kx_3 + \kappa c^\dagger + \bar{\kappa} c$.

We have:

$$\begin{aligned}
(a_A a_B)^{11} &= \Big(\zeta_A(\omega,k)\zeta_B(\omega',k')e^{i(k+k')x_3} + \\
&\qquad + \zeta_A(\omega,m,n)\bar{\zeta}_B(\omega',m,l)H_n(x_3)H_l(x_3)e^{-2x_3^2}\Big),\\
(a_A a_B)^{13}_m &= H_n(x_3)\exp -(ikx_3 + x_3^2)\times\\
&\quad\times\Big(\zeta_A(\omega,k)\zeta_B(\omega',m,n) + \zeta_A(\omega,\bar{m},n)\zeta_B(\omega',\vec{k})\langle \bar{m}{-}1|e^{-i(\kappa c^\dagger + \bar{\kappa} c)}|m{-}1\rangle\Big)\,,\\
(a_A a_B)^{31}_m &= \overline{(a_B a_A)^{13}_m}\,,\\
(a_A a_B)^{33}_{m\bar{m}} &= \bar{\zeta}_A(\omega,m,l)\zeta_B(\omega',\bar{m},p)H_l(x_3)H_p(x_3)e^{-2x_3^2} + \\
&\quad + \zeta_A(\omega,\vec{k})\zeta_B(\omega',\vec{k}')e^{-\frac{i}{2}\vec{k}\times\vec{k}'}\langle m{-}1|e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}|\bar{m}{-}1\rangle\,.
\end{aligned} \tag{5.2}$$

5.1 The cubic interactions

Let us look at the cubic term $\mathcal{L}_3$ in the expansion of the Yang-Mills lagrangian about the fluxon solution. It, of course, contains the usual interactions of the bulk photons and their superpartners. More interesting is that it contains interactions of the modes from the V_{11} sector, i.e. the fluctuations of the string, with the V_{13}, V_{31} modes. In turn, these modes can annihilate into bulk modes.

We shall also need the expressions for $D^0_B a_A$:

$$\begin{aligned}
D^0_t a_A(\omega) &= i\omega a_A(\omega)\\
D^0_3 a_A^{11,33} &= -ik a_A^{11,33}\\
D^0_3 a_A^{13} &= \zeta_A(\omega,m,n)\left(\sqrt{n}H_{n-1}(x_3) - \sqrt{n+1}H_{n+1}(x_3)\right)e^{i\omega t - x_3^2}|0\rangle\langle m|\\
D^0_4 a_A^{13} &= -i\zeta_A(\omega,m,n)\left(\sqrt{n}H_{n-1}(x_3) + \sqrt{n+1}H_{n+1}(x_3)\right)e^{i\omega t - x_3^2}|0\rangle\langle m|\\
\bar{D}^0 a_A^{13} &= \zeta_A(\omega,m,n)H_n(x_3)e^{i\omega t - x_3^2}\sqrt{m-1}|0\rangle\langle m-1|\\
D^0 a_A^{13} &= -\zeta_A(\omega,m,n)H_n(x_3)e^{i\omega t - x_3^2}\sqrt{m}|0\rangle\langle m+1|\\
D^0 a_A^{33} &= -i\bar{\kappa}\zeta_A(\omega,\vec{k})\,S\,e^{i(\omega t - \vec{k}\cdot\vec{x})}\,S^\dagger\,,
\end{aligned} \tag{5.3}$$

as well as the overlap integrals/matrix elements:

$$\begin{aligned}
\mathcal{V}_{\|}(k,n,\bar{n}) &= \int dx\,e^{-ikx-2x^2}H_{\bar{n}}(x)H_n(x)\,,\\
\mathcal{V}_{\perp}(\kappa,n,\bar{n}) &= \langle \bar{n}|e^{-i(\bar{\kappa}c+\kappa c^\dagger)}|n\rangle\,.
\end{aligned} \tag{5.4}$$

The integrals (5.4) are easy to evaluate using the oscillator representation:

$$e^{-x_3^2} H_n(x_3) \sim |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\,, \qquad a = \left(\frac{1}{2}\partial_3 + x_3\right), a^\dagger = \left(-\frac{1}{2}\partial_3 + x_3\right).$$

Then:

$$\begin{aligned}
\mathcal{V}_{||}(k,n,\bar{n}) &= \langle \bar{n}| e^{-\frac{ik}{2}(a+a^\dagger)}|n\rangle \\
&= e^{-\frac{k^2}{8}} \sum_{m\geq 0} \left(\frac{ik}{2}\right)^{\bar{n}+n-2m} \frac{\sqrt{n!\bar{n}!}}{m!(n-m)!(\bar{n}-m)!}\,, \\
\mathcal{V}_{\perp}(\kappa,n,\bar{n}) &= e^{-\frac{\kappa\bar{\kappa}}{2}} \sum_{m\geq 0} \frac{\sqrt{n!\bar{n}!}\,(-i\kappa)^{\bar{n}-m}(-\bar{\kappa})^{n-m}}{m!(n-m)!(\bar{n}-m)!}\,.
\end{aligned} \tag{5.5}$$

The three-point disc amplitudes are labelled by the types of the D-brane boundary conditions one imposes between the points of operator insertions. Up to cyclic permutations we have the following options:

$$1-1-1\,,\ 1-1-3\,,\ 1-3-3\,,\ 3-3-3\,.$$

They correspond to the triple vertices between the following modes in the gauge theory:

$$a^{11}a^{11}a^{11}\,,\ a^{11}a^{13}a^{31}\,,\ a^{13}a^{33}a^{31}\,,\ a^{33}a^{33}a^{33}\,,$$

respectively. We shall look at the bosonic vertex only, the $\lambda\lambda a$ vertices follow by supersymmetry.

The term $\mathcal{L}_3$ being proportional to commutators will vanish for $N=1$ in the $1-1-1$ sector.

For other sectors it is convenient to rewrite the $\mathrm{Tr}\, D_B^0 a_C [a_B, a_C]$ term as

$$2\,\mathrm{Tr}(D_B^0 a_C) a_B a_C\,,$$

using the gauge condition (4.4).

After straightforward computation we arrive at the following amplitudes

$$\begin{aligned}
\mathcal{A} &= \delta(\omega_1+\omega_2-\omega_3)\mathcal{A}^{abc}(\omega_1,\omega_2,\omega_3) \\
\mathcal{A}^{113} &= 2i\mathcal{V}_{||}(k_1,n,l)\,\times \\
&\quad \times\Big[\omega_1\,\zeta_A(\omega_1,k_1)\left(\bar{\zeta}_A(\omega_3,m,l)\zeta_t(\omega_2,m,n) - \zeta_A(\omega_2,m,n)\bar{\zeta}_t(\omega_3,m,l)\right) - \\
&\qquad - k_1\,\zeta_A(\omega_1,k_1)\left(\bar{\zeta}_A(\omega_3,m,l)\zeta_3(\omega_2,m,n) - \zeta_A(\omega_2,m,n)\bar{\zeta}_3(\omega_3,m,l)\right) - \\
&\qquad - \omega_2\zeta_t(\omega_1,k_1)\zeta_A(\omega_2,m,n)\bar{\zeta}_A(\omega_3,m,l) + \\
&\qquad + i\sqrt{m}\,(\zeta_1+i\zeta_2)\,(\omega_1,k_1)\zeta_A(\omega_2,m+1,n)\bar{\zeta}_A(\omega_3,m,l) - \\
&\qquad - i\sqrt{m}\,(\zeta_1-i\zeta_2)\,(\omega_1,k_1)\zeta_A(\omega_2,m,n)\bar{\zeta}_A(\omega_3,m+1,l) - \\
&\qquad - i\sqrt{n+1}\,(\zeta_3+i\zeta_4)\,(\omega_1,k_1)\zeta_A(\omega_2,m,n+1)\bar{\zeta}_A(\omega_3,m,l) + \\
&\qquad + i\sqrt{n}\,(\zeta_3+i\zeta_4)\,(\omega_1,k_1)\zeta_A(\omega_2,m,n-1)\bar{\zeta}_A(\omega_3,m,l)\Big]
\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}^{133} = & \, 2i\mathcal{V}_{\|}(k_1,n,l)\mathcal{V}_{\perp}(\kappa_1,m-1,\bar{m}-1)\times \\
& \times\Big[\zeta_A(\omega_1,\vec{k}_1)k_1^\mu\left(\bar{\zeta}_\mu(\omega_2,m,l)\zeta_A(\omega_3,\bar{m},n)-\zeta_\mu(\omega_3,\bar{m},p)\bar{\zeta}_A(\omega_2,m,l)\right)- \\
& \quad -\omega_2\zeta_t(\omega_1,\vec{k}_1)\zeta_A(\omega_2,\bar{m},n)\bar{\zeta}_A(\omega_3,m,l)+ \\
& \quad +i\sqrt{\bar{m}}\,(\zeta_1+i\zeta_2)\,(\omega_1,\vec{k}_1)\zeta_A(\omega_2,\bar{m}+1,n)\bar{\zeta}_A(\omega_3,m,l)- \\
& \quad -i\sqrt{\bar{m}-1}\,(\zeta_1-i\zeta_2)\,(\omega_1,\vec{k}_1)\zeta_A(\omega_2,\bar{m}-1,n)\bar{\zeta}_A(\omega_3,m,l)- \\
& \quad -i\sqrt{n}\,(\zeta_3+i\zeta_4)\,(\omega_1,\vec{k}_1)\zeta_A(\omega_2,\bar{m},n-1)\bar{\zeta}_A(\omega_3,m,l)- \\
& \quad -i\sqrt{n+1}\,(\zeta_3-i\zeta_4)\,(\omega_1,\vec{k}_1)\zeta_A(\omega_2,\bar{m},n+1)\bar{\zeta}_A(\omega_3,m,l)\big]\,.
\end{aligned}
\tag{5.6}
$$

Finally, the bulk + bulk$\to$ bulk amplitude coincides with that of the ordinary U(1) noncommutative gauge theory:

$$
\mathcal{A}^{333} = -\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\sin\left(-\frac{1}{2}\vec{k}_2\times\vec{k}_3\right)k_1^\mu\zeta_A(\omega_1,\vec{k}_1)\zeta_\mu(\omega_2,\vec{k}_2)\zeta_A(-\omega_3,\vec{k}_3)\,. \tag{5.7}
$$

5.2 The quartic interactions

The quartic vertices arise from the term

$$
S_4 = -\frac{\pi g^2}{2}\int \mathrm{d}t\mathrm{d}x_3\,\mathrm{Tr}\left(a_Aa_Ba_Aa_B-a_A^2a_B^2\right). \tag{5.8}
$$

We substitute (4.24) into (5.8) to get the vertices for the quartic interactions between the various quanta ζ_A. Basically all we need to do is to multiply (5.1) by (5.1), shuffle the indices around and to take the trace over $\mathcal{H}$. The resulting expression is rather messy. We present here only the amplitudes which are responsible for

$$
11+11\to 13+31
$$

process and those related to it by crossing: $\delta(\omega+\omega'+\tilde{\omega}+\tilde{\omega}')\mathcal{A}_4(\omega,\ldots)$, where:

$$
\begin{aligned}
\mathcal{A}_4 = & \left[\zeta_A(\tilde{\omega},\tilde{n},m)\bar{\zeta}_B(-\tilde{\omega}',\tilde{n}',m)-\bar{\zeta}_A(-\tilde{\omega},\tilde{n},m)\zeta_B(\tilde{\omega}',\tilde{n}',m)\right]\times \\
& \times\zeta_A(\omega,k)\zeta_B(\omega',k')\mathcal{V}_{113}(k+k',\tilde{n},\tilde{n}')+ \\
& +\Big(\left[\zeta_A(\omega,k)\zeta_B(\omega',n,m)-A\leftrightarrow B\right]\zeta_A(\tilde{\omega},\tilde{k})\bar{\zeta}_B(-\tilde{\omega}',\tilde{n},m)+ \\
& \quad +\left[\zeta_A(\omega,k)\bar{\zeta}_B(-\omega',n,m)-A\leftrightarrow B\right]\zeta_A(\tilde{\omega},\tilde{k})\zeta_B(\tilde{\omega}',\tilde{n},m)\Big)\mathcal{V}_{113}(k+\tilde{k},n,\tilde{n})\,.
\end{aligned}
\tag{5.9}
$$

6. Fluctuations and interactions of the N-fluxon

In this section we shall briefly discuss the generalization of the above discussion of the fluctuations and interactions of the fluxon to the case of the N-fluxon. In this case the Hilbert space is decomposed into

$$
\mathcal{H} = V_N\oplus\mathcal{H}_N,\quad V_N = P_N\mathcal{H},\ \mathcal{H}_N = (1-P_N)\mathcal{H}\,, \tag{6.1}
$$

and the background fields satisfy $A_4^0 = i\Phi^0$, $F_{1,2}^0$, $F_{13,4}^0 \propto P_N$ and D, $\bar{D} \in \mathcal{H}_N$. Thus, as before, we shall write the space End$(\mathcal{H})$ of the operators in the Fock space as a direct sum of subspaces:

$$\mathrm{End}\,\mathcal{H} = V_{11} \oplus V_{13} \oplus V_{31} \oplus V_{33}\,, \tag{6.2}$$

where

$$V_{11} = V_N \otimes V_N\,, \qquad V_{13} = V_N \otimes \mathcal{H}_N\,, \qquad V_{31} = \mathcal{H}_N \otimes V_N\,, \qquad V_{33} = \mathcal{H}_N \hat{\otimes} \mathcal{H}_N\,. \tag{6.3}$$

Let us consider the 1-1 modes that are described by $A_B = A_B^0 + g a_B$, where $a_B \in V_{11}$, and write these as

$$a_B = \frac{1}{\sqrt{\theta}} \sum_{i,j=1}^{N} \mathcal{A}_B^{ij} |i\rangle\langle j|\,, \qquad B = 0, 1, \ldots 9\,, \quad \mathcal{A}_B = [\mathcal{A}_B^{ij}]\,. \tag{6.4}$$

Notice that we have normalized $\mathcal{A}_B$ to be dimensionless, as befits a two-dimensional gauge field. Then, using the fact that $[D, a_B] = [\bar{D}, a_B] = [F_{AB}, a_C] = 0$, we easily find that:

$$\mathcal{S} = \int dt dx_3 \left[\frac{2\pi}{g^2\theta} + \mathrm{Tr}\left(\frac{1}{2}\mathcal{A}_B(\partial_t^2 - \partial_3^2)\mathcal{A}_B + \frac{g}{\sqrt{\theta}}\partial_\alpha \mathcal{A}_B[\mathcal{A}_\alpha, \mathcal{A}_B] + \frac{g^2}{\theta}[\mathcal{A}_B, \mathcal{A}_C]^2\right)\right] + \\ + \text{fermions}\,, \tag{6.5}$$

where $\alpha = 0, 3$ and we are working in the gauge $\partial_t \mathcal{A}_0 - \partial_3 \mathcal{A}_3 = 0$. Thus, as expected, the 1-1 fluctuations about the N-fluxon are described by the 2 dimensional U(N) $\mathcal{N} = 8$ super-Yang-Mills theory, the dimensionally reduced d=10 supersymmetric gauge theory, with coupling $g/\sqrt{\theta}$.

If we were to perturb about the separated N-fluxon (3.26), the above action for the 1-1 modes would be modifed, since now $\Phi = -2x_3 P_N + D_N$ is no longer proportional to the identity in V_N. In particular, we will generate from the quartic term in the action a mass term for the 1-1 modes equal to $\sum_{ij}(d_i - d_j)^2 \mathcal{A}_{ij}^2$, as befits the low energy modes of fundamental strings on separated D1-branes.

In addition, one can easily derive, following the steps described above for the 1-fluxon, the spectrum and interactions of the 1-3, 3-1, and 3-3 modes. Note that the 1-3 and 3-1, discrete energy, modes will now be in the fundamental representation of U(N).

7. Dyons, ϑ-angles

So far we have been studying magnetic flux tubes, which were obtained by solving the Bogomolny equation

$$[D_i, \Phi] + B_i = 0\,.$$

Let us look at more general solutions, with both electric and magnetic charges. The appropriate BPS equations are gotten by using the noncommutative analogue of the usual BPS inequalities:

$$
\begin{aligned}
g^2\vec{E}^2+\frac{1}{g^2}\vec{B}^2+\frac{1}{g^2}(\vec{D}\Phi)^2 &= \left(g\vec{E}+\frac{\sin\alpha}{g}\,\vec{D}\Phi\right)^2+\frac{1}{g^2}(\vec{B}+\cos\alpha\,\vec{D}\Phi)^2-\\
&\quad -\left(\vec{E}\sin\alpha+\frac{1}{g^2}\vec{B}\cos\alpha\right)\cdot\vec{D}\Phi-\\
&\quad -\vec{D}\Phi\cdot\left(\vec{E}\sin\alpha+\frac{1}{g^2}\vec{B}\cos\alpha\right)\\
&\geq -\vec{D}\cdot\left(\Phi\star\left(\sin\alpha\,\vec{E}+\frac{\cos\alpha}{g^2}\,\vec{B}\right)+\right.\\
&\qquad\left.+\left(\sin\alpha\,\vec{E}+\frac{\cos\alpha}{g^2}\,\vec{B}\right)\star\Phi\right).
\end{aligned}
\tag{7.1}
$$

They are:

$$
\begin{aligned}
\cos\alpha\,[D_i,\Phi]+B_i &= 0\,,\\
\sin\alpha\,[D_i,\Phi]+g^2E_i &= 0\,,
\end{aligned}
\tag{7.2}
$$

where $E_i=-i\frac{F_{0i}}{g^2}$ is the electric field. They are easy to solve: take A_i to be equal to the monopole solution gauge field A_i^{mon} found in [11], or to the q-fluxon solution A^{flux} discussed in this paper, and then take

$$
\begin{aligned}
\Phi &= \frac{\Phi^{\text{mon}}}{\cos\alpha}\quad\text{or}\quad\frac{\Phi^{\text{flux}}}{\cos\alpha}\,,\\
A_t &= -i\tan\alpha\,\Phi^{\text{mon}}\quad\text{or}\quad -i\tan\alpha\,\Phi^{\text{flux}}\,,\quad A_3=0\,.
\end{aligned}
\tag{7.3}
$$

In this way one gets a solution with both electric and magnetic charges:

$$
g^2\,Q_e=\tan\alpha\,Q_m=q\tan\alpha\,.
\tag{7.4}
$$

The magnetic field will be as before (dyons in the noncommutative Yang-Mills theory were also recently discussed in [14], note the 2π factor difference in our normalization of electric charge compared to that reference).

Consider the case of the q-fluxon. The only nonvanishing component of the electric field will be

$$
g^2E_3=\tan\alpha B_3\,,
\tag{7.5}
$$

so that we have a flux tube of magnetic and electric field along the x_3 axis. The tension of this dyonic fluxon will be

$$
T=\frac{2q\pi}{g^2\theta(\cos\alpha)^2}\,.
\tag{7.6}
$$

Of course, in the quantum theory Q_e must be an integer. This will emerge in the standard fashion once we quantize the solitons. This will fix the allowed values of α so that

$$\tan\alpha = g^2 \frac{p}{q}\,, \qquad p = 1, 2, 3, \dots$$

and the tension will be

$$T = \frac{2\pi}{q\theta}\left(\frac{q^2}{g^2} + g^2 p^2\right). \tag{7.7}$$

This can be seen by transforming the above solution to $A_t = 0$ gauge, wherein $D, \bar{D}$, are unchanged and

$$A_t = 0\,, \qquad \Phi = -2x_3 P_N\,, \qquad A_3 = -2it \times \tan\alpha P_N\,. \tag{7.8}$$

Let us briefly discuss how this formula is consistent with the familiar S-duality covariant expression for the tension of the (p, q) string:

$$T_{(p,q)} = \frac{1}{\alpha'}\sqrt{\frac{q^2}{(2\pi g_s)^2} + p^2}\,. \tag{7.9}$$

In the presence of the ϑ-angle the relation between the electric and magnetic charges is modified. The ϑ-angle is the vev of the ten-dimensional axion field, which makes the dilaton a complex field. It will effectively shift p by ϑq in the formula above. The same effect is present in the gauge theory (Witten's effect). Let us therefore set $\vartheta = 0$ for clarity.

Recall [6] the relation between the closed string background and the parameters of the noncommutative gauge theory. We have the B-field $\frac{1}{2}B\mathrm{d}x^1 \wedge \mathrm{d}x^2$, closed string coupling g_s which entered (7.9):

$$g_s = \frac{g^2\alpha'}{\sqrt{(2\pi\alpha')^2 + \theta^2}}\,, \qquad B = \frac{\theta}{(2\pi\alpha')^2 + \theta^2}\,. \tag{7.10}$$

Recall that the B-field tilts the D-string towards the D3-brane [15, 16, 11]. From our formulae for the scalar field (7.3) and the Dirac-Born-Infeld intuition as in [11] we conclude that the (p, q)-string forms an angle $\psi_{(p,q)}$ with the D3-brane:

$$\tan\psi_{(p,q)} = \frac{2\pi\alpha'}{\theta\cos\alpha}\,. \tag{7.11}$$

The tension of the projection onto the D3-brane of the tilted (p, q)-string is equal to:

$$\frac{T_{(p,q)}}{\cos\psi_{(p,q)}} = \frac{2\pi}{g^2\theta q}\left(q^2 + g^4p^2\right) + \Delta T\,, \tag{7.12}$$

with

$$\Delta T = \frac{q\theta}{2\pi(g\alpha')^2}$$

being the defect. Notice that this defect term depends only on the magnetic charge. It is q times higher then the analogous defect term computed in [11] where it was interpreted as the work done by the magnetic force in bringing the semi-infinite string in from infinity. We should imagine constructing our dyonic string by taking the semi-infinite string, the (p,q)-version of the solution of [11], and then translating it as in the section 3.2. Although the endpoint of the (p,q) string is a (p,q) dyon, the background space-like B field couples only to the magnetic charge, i.e. only to q. Upon subtracting this defect we precisely match the gauge theory answer (7.7) and the tension expected from the D-brane consideration.

8. Conclusions

In this paper we have constructed a large class of very simple BPS solutions of $d = 4$, supersymmetric noncommutative Yang-Mills theory — that correspond to D1-strings intersecting D3-branes in the presence of a background B-field in the decoupling limit. We analysed in some detail the fluctuations about these solitons and their interactions. The results were in complete agreement with the expecteations from string theory. In particular we found the fluctuations of the superstring in 10 dimensions arising from fundamental strings attached to the D1-strings, the ordinary particles of the gauge theory in 4 dimensions and a set of states with discrete spectrum, localized at the intersection point, corresponding to fundamental strings stretched between the D1 string and the D3-brane.

Unfortunately, in the semiclassical treatment that we have employed we are unable to see the massive modes of the D1-string, whose energies are of order $1/g^2$. Perhaps some of these modes are visable as BPS states that can be constructed as gauge theory solitons.

There are many facets of our investigation which remain to be completed; including the explicit construction of noncommutative monopole strings for noncommutative $\mathrm{U}(N)$ theory and the generalization of our considerations to other branes and dimensions. For example, it is easy to use of construction of fluxons to construct vortex solitons of 1+2 dimensional noncommutative gauge theory. Simply take our solution for $\bar{D}$ and D, (3.22), and throw away the Φ field.

Finally, the implications of these solitons for the dynamics of large N-gauge theory remain to be investigated.

Acknowledgments

We would like to thank A. Hashimoto, N. Itzhaki, G. Moore, Polchinski and E. Witten for discussions. Our research was partially supported by NSF under the grants PHY94-07194 and PHY 97-22022; in addition, research of N.N. was supported by Robert H. Dicke fellowship from Princeton University, partly by RFFI under grant 00-02-16530, partly by the grant 00-15-96557 for scientific schools. N.N. is grateful to ITP, UC Santa Barbara and IHES at Bures-sur-Yvette, for their hospitality during various stages of this work.

JHEP10(2000)021

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